11.6 Areas of Regular Polygons

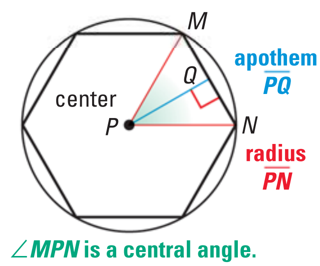
Before You found areas of circles.

Now You will find areas of regular polygons inscribed in circles.

Why? So you can understand the structure of a honeycomb, as in Ex. 44.

Key Vocabulary
- center of a polygon
- radius of a polygon
- apothem of a polygon
- central angle of a regular polygon

The diagram shows a regular polygon inscribed in a circle. The **center of the polygon** and the **radius of the polygon** are the center and the radius of its circumscribed circle.

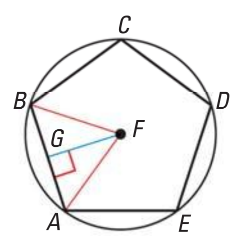


The distance from the center to any side of the polygon is called the **apothem of the polygon**. The apothem is the height to the base of an isosceles triangle that has two radii as legs.

A **central angle of a regular polygon** is an angle formed by two radii drawn to consecutive vertices of the polygon. To find the measure of each central angle, divide 360° by the number of sides.

EXAMPLE 1 Find angle measures in a regular polygon

In the diagram, *ABCDE* is a regular pentagon inscribed in ⊙*F*. Find each angle measure.



a. $m\angle AFB$ **b.** $m\angle AFG$ **c.** $m\angle GAF$

Solution

a. $\angle AFB$ is a central angle, so $m\angle AFB = \frac{360°}{5}$, or 72°.

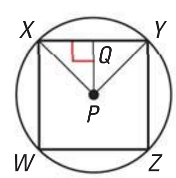
b. $\overline{FG}$ is an apothem, which makes it an altitude of isosceles $\triangle AFB$.
So, $\overline{FG}$ bisects $\angle AFB$ and $m\angle AFG = \frac{1}{2} m\angle AFB = 36°$.

c. The sum of the measures of right $\triangle GAF$ is 180°.
So, $90° + 36° + m\angle GAF = 180°$, and $m\angle GAF = 54°$.

READ DIAGRAMS
A segment whose length is *the apothem* is sometimes called *an apothem*. The segment is an altitude of an isosceles triangle, so it is also a median and angle bisector of the isosceles triangle.

✓ GUIDED PRACTICE for Example 1

In the diagram, *WXYZ* is a square inscribed in ⊙*P*.



1. Identify the center, a radius, an apothem, and a central angle of the polygon.

2. Find $m\angle XPY$, $m\angle XPQ$, and $m\angle PXQ$.

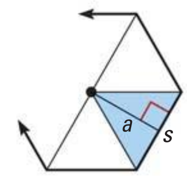
AREA OF AN *n*-GON You can find the area of any regular *n*-gon by dividing it into congruent triangles.

A = Area of one triangle • Number of triangles

$= \left(\frac{1}{2} \cdot s \cdot a\right) \cdot n$ **Base of triangle is *s* and height of triangle is *a*. Number of triangles is *n*.**

$= \frac{1}{2} \cdot a \cdot (n \cdot s)$ **Commutative and Associative Properties of Equality**

$= \frac{1}{2}a \cdot P$ **There are *n* congruent sides of length *s*, so perimeter *P* is *n* • *s*.**



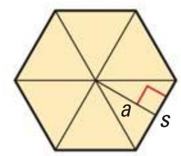
READ DIAGRAMS

In this book, a point shown inside a regular polygon marks the center of the circle that can be circumscribed about the polygon.

THEOREM *For Your Notebook*

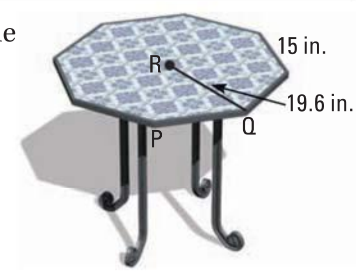
THEOREM 11.11 Area of a Regular Polygon

The area of a regular *n*-gon with side length *s* is half the product of the apothem *a* and the perimeter *P*, so $A = \frac{1}{2}aP$, or $A = \frac{1}{2}a \cdot ns$.



EXAMPLE 2 Find the area of a regular polygon

DECORATING You are decorating the top of a table by covering it with small ceramic tiles. The table top is a regular octagon with 15 inch sides and a radius of about 19.6 inches. What is the area you are covering?



Solution

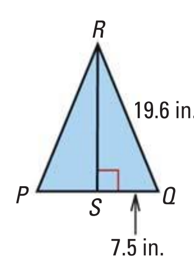
STEP 1 **Find** the perimeter P of the table top.
An octagon has 8 sides, so $P = 8(15) = 120$ inches.

STEP 2 **Find** the apothem a. The apothem is height RS of $\triangle PQR$.
Because $\triangle PQR$ is isosceles, altitude $\overline{RS}$ bisects $\overline{QP}$.

So, $QS = \frac{1}{2}(QP) = \frac{1}{2}(15) = 7.5$ inches.

To find RS, use the Pythagorean Theorem for $\triangle RQS$.

$a = RS \approx \sqrt{19.6^2 - 7.5^2} = \sqrt{327.91} \approx 18.108$



STEP 3 **Find** the area A of the table top.

$A = \frac{1}{2}aP$ **Formula for area of regular polygon**

$\approx \frac{1}{2}(18.108)(120)$ **Substitute.**

≈ 1086.5 **Simplify.**

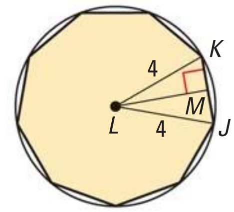
▶ So, the area you are covering with tiles is about 1086.5 square inches.

ROUNDING

In general, your answer will be more accurate if you avoid rounding until the last step. Round your final answers to the nearest tenth unless you are told otherwise.

EXAMPLE 3 Find the perimeter and area of a regular polygon

A regular nonagon is inscribed in a circle with radius 4 units. Find the perimeter and area of the nonagon.



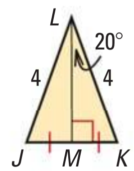
Solution

The measure of central $\angle JLK$ is $\frac{360°}{9}$, or 40°. Apothem $\overline{LM}$ bisects the central angle, so $m\angle KLM$ is 20°. To find the lengths of the legs, use trigonometric ratios for right $\triangle KLM$.

$$\sin 20° = \frac{MK}{LK} \qquad \cos 20° = \frac{LM}{LK}$$

$$\sin 20° = \frac{MK}{4} \qquad \cos 20° = \frac{LM}{4}$$

$$4 \cdot \sin 20° = MK \qquad 4 \cdot \cos 20° = LM$$



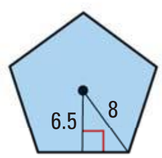
The regular nonagon has side length $s = 2MK = 2(4 \cdot \sin 20°) = 8 \cdot \sin 20°$ and apothem $a = LM = 4 \cdot \cos 20°$.

▶ So, the perimeter is $P = 9s = 9(8 \cdot \sin 20°) = 72 \cdot \sin 20° \approx 24.6$ units, and the area is $A = \frac{1}{2}aP = \frac{1}{2}(4 \cdot \cos 20°)(72 \cdot \sin 20°) \approx 46.3$ square units.

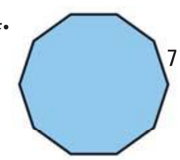
✓ GUIDED PRACTICE for Examples 2 and 3

Find the perimeter and the area of the regular polygon.

3.



4.



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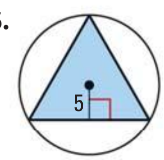


6. Which of Exercises 3–5 above can be solved using special right triangles?

CONCEPT SUMMARY *For Your Notebook*

Finding Lengths in a Regular *n*-gon

To find the area of a regular *n*-gon with radius *r*, you may need to first find the apothem ***a*** or the side length ***s***.

You can use . . .	. . . when you know *n* and . . .	. . . as in . . .
Pythagorean Theorem: $\left(\frac{1}{2}s\right)^2 + a^2 = r^2$	Two measures: *r* and ***a***, or *r* and ***s***	Example 2 and Guided Practice Ex. 3.
Special Right Triangles	Any one measure: *r* or ***a*** or ***s*** **And** the value of *n* is 3, 4, or 6	Guided Practice Ex. 5.
Trigonometry	Any one measure: *r* or ***a*** or ***s***	Example 3 and Guided Practice Exs. 4 and 5.

11.6 EXERCISES

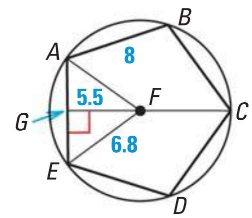
HOMEWORK KEY

○ = **WORKED-OUT SOLUTIONS** on p. WS1 for Exs. 7, 21, and 37

★ = **STANDARDIZED TEST PRACTICE** Exs. 5, 18, 22, and 44

SKILL PRACTICE

VOCABULARY **In Exercises 1–4, use the diagram shown.**



1. Identify the *center* of regular polygon *ABCDE*.
2. Identify a *central angle* of the polygon.
3. What is the *radius* of the polygon?
4. What is the *apothem*?

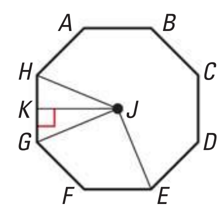
5. ★ **WRITING** *Explain* how to find the measure of a *central angle* of a regular polygon with *n* sides.

EXAMPLE 1 on p. 762 for Exs. 6–13

MEASURES OF CENTRAL ANGLES **Find the measure of a central angle of a regular polygon with the given number of sides. Round answers to the nearest tenth of a degree, if necessary.**

6. 10 sides
7. 18 sides
8. 24 sides
9. 7 sides

FINDING ANGLE MEASURES **Find the given angle measure for the regular octagon shown.**

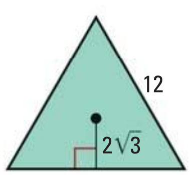


10. $m\angle GJH$
11. $m\angle GJK$
12. $m\angle KGJ$
13. $m\angle EJH$

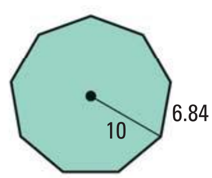
EXAMPLE 2 on p. 763 for Exs. 14–17

FINDING AREA **Find the area of the regular polygon.**

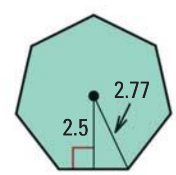
14.



15.



16.



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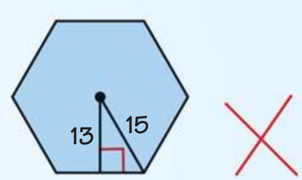
17. **ERROR ANALYSIS** *Describe* and correct the error in finding the area of the regular hexagon.

$\sqrt{15^2 - 13^2} \approx 7.5$

$A = \frac{1}{2}a \cdot ns$

$A = \frac{1}{2}(13)(6)(7.5) = 292.5$



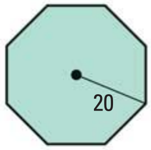
EXAMPLE 3 on p. 764 for Exs. 18–25

18. ★ **MULTIPLE CHOICE** Which expression gives the apothem for a regular dodecagon with side length 8?

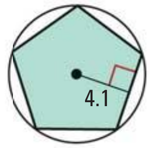
(A) $a = \frac{4}{\tan 30°}$ (B) $a = \frac{4}{\tan 15°}$ (C) $a = \frac{8}{\tan 15°}$ (D) $a = 8 \cdot \cos 15°$

PERIMETER AND AREA **Find the perimeter and area of the regular polygon.**

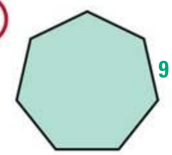
19.



20.



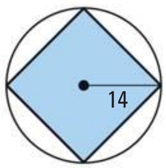
21.



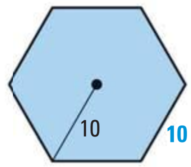
22. ★ **SHORT RESPONSE** The perimeter of a regular nonagon is 18 inches. Is that enough information to find the area? If so, find the area and *explain* your steps. If not, *explain* why not.

CHOOSE A METHOD **Identify any unknown length(s) you need to know to find the area of the regular polygon. Which methods in the table on page 764 can you use to find those lengths? Choose a method and find the area.**

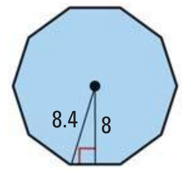
23.



24.



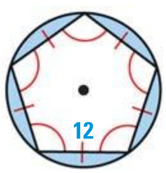
25.



26. **INSCRIBED SQUARE** Find the area of the *unshaded* region in Exercise 23.

POLYGONS IN CIRCLES **Find the area of the shaded region.**

27.



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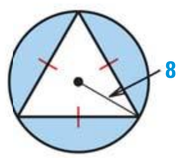
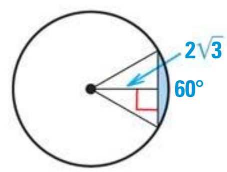


29.



30. **COORDINATE GEOMETRY** Find the area of a regular pentagon inscribed in a circle whose equation is given by $(x - 4)^2 + (y + 2)^2 = 25$.

REASONING **Decide whether the statement is *true* or *false*. *Explain*.**

31. The area of a regular n-gon of fixed radius r increases as n increases.

32. The apothem of a regular polygon is always less than the radius.

33. The radius of a regular polygon is always less than the side length.

34. **FORMULAS** In Exercise 44 on page 726, the formula $A = \frac{\sqrt{3}s^2}{4}$ for the area A of an equilateral triangle with side length s was developed. Show that the formulas for the area of a triangle and for the area of a regular polygon, $A = \frac{1}{2}bh$ and $A = \frac{1}{2}a \cdot ns$, also result in this formula when they are applied to an equilateral triangle with side length s.

35. **CHALLENGE** An equilateral triangle is shown inside a square inside a regular pentagon inside a regular hexagon. Write an expression for the exact area of the shaded regions in the figure. Then find the approximate area of the entire shaded region, rounded to the nearest whole unit.

PROBLEM SOLVING

EXAMPLE 3
on p. 764
for Ex. 36

36. BASALTIC COLUMNS Basaltic columns are geological formations that result from rapidly cooling lava. The Giant's Causeway in Ireland, pictured here, contains many hexagonal columns. Suppose that one of the columns is in the shape of a regular hexagon with radius 8 inches.

a. What is the apothem of the column?

b. Find the perimeter and area of the column. Round the area to the nearest square inch.

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37. WATCH A watch has a circular face on a background that is a regular octagon. Find the apothem and the area of the octagon. Then find the area of the silver border around the circular face.

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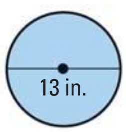
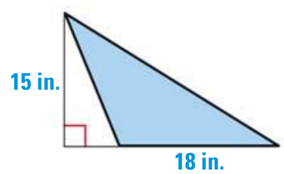
38. COMPARING AREAS *Predict* which figure has the greatest area and which has the smallest area. Check by finding the area of each figure.

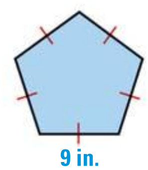
a.



b.



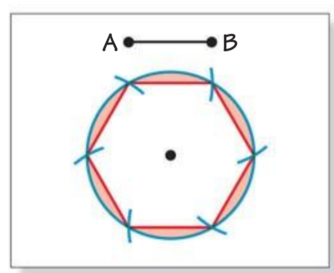
c.



39. CRAFTS You want to make two wooden trivets, a large one and a small one. Both trivets will be shaped like regular pentagons. The perimeter of the small trivet is 15 inches, and the perimeter of the large trivet is 25 inches. Find the area of the small trivet. Then use the Areas of Similar Polygons Theorem to find the area of the large trivet. Round your answers to the nearest tenth.

40. CONSTRUCTION Use a ruler and compass.

a. Draw $\overline{AB}$ with a length of 1 inch. Open the compass to 1 inch and draw a circle with that radius. Using the same compass setting, mark off equal parts along the circle. Then connect the six points where the compass marks and circle intersect to draw a regular hexagon as shown.



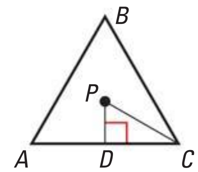
b. What is the area of the hexagon? of the shaded region?

c. *Explain* how to construct an equilateral triangle.

41. HEXAGONS AND TRIANGLES Show that a regular hexagon can be divided into six equilateral triangles with the same side length.

42. ALTERNATIVE METHODS Find the area of a regular hexagon with side length 2 and apothem $\sqrt{3}$ in at least four different ways.

43. **APPLYING TRIANGLE PROPERTIES** In Chapter 5, you learned properties of special segments in triangles. Use what you know about special segments in triangles to show that radius CP in equilateral $\triangle ABC$ is twice the apothem DP.



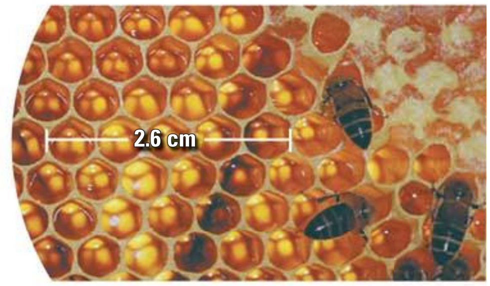
44. ★ **EXTENDED RESPONSE** Assume that each honeycomb cell is a regular hexagon. The distance is measured through the center of each cell.

a. Find the average distance across a cell in centimeters.

b. Find the area of a "typical" cell in square centimeters. Show your steps.

c. What is the area of 100 cells in square centimeters? in square decimeters? (1 decimeter = 10 centimeters.)

d. Scientists are often interested in the number of cells per square decimeter. *Explain* how to rewrite your results in this form.



45. **CONSTANT PERIMETER** Use a piece of string that is 60 centimeters long.

a. Arrange the string to form an equilateral triangle and find the area. Next form a square and find the area. Then do the same for a regular pentagon, a regular hexagon, and a regular decagon. What is happening to the area?

b. Predict and then find the areas of a regular 60-gon and a regular 120-gon.

c. Graph the area A as a function of the number of sides n. The graph approaches a limiting value. What shape do you think will have the greatest area? What will that area be?

46. **CHALLENGE** Two regular polygons both have n sides. One of the polygons is inscribed in, and the other is circumscribed about, a circle of radius r. Find the area between the two polygons in terms of n and r.

MIXED REVIEW

PREVIEW
Prepare for Lesson 11.7 in Exs. 47–51.

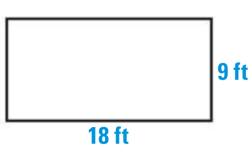
A jar contains 10 red marbles, 6 blue marbles, and 2 white marbles. Find the probability of the event described. *(p. 893)*

47. You randomly choose one red marble from the jar, put it back in the jar, and then randomly choose a red marble.

48. You randomly choose one blue marble from the jar, keep it, and then randomly choose one white marble.

Find the ratio of the width to the length of the rectangle. Then simplify the ratio. *(p. 356)*

49.



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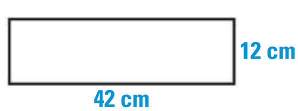

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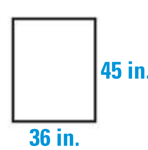

52. The vertices of quadrilateral $ABCD$ are $A(-3, 3)$, $B(1, 1)$, $C(1, -3)$, and $D(-3, -1)$. Draw $ABCD$ and determine whether it is a parallelogram. *(p. 522)*

EXTRA PRACTICE for Lesson 11.6, p. 917 **ONLINE QUIZ** at classzone.com